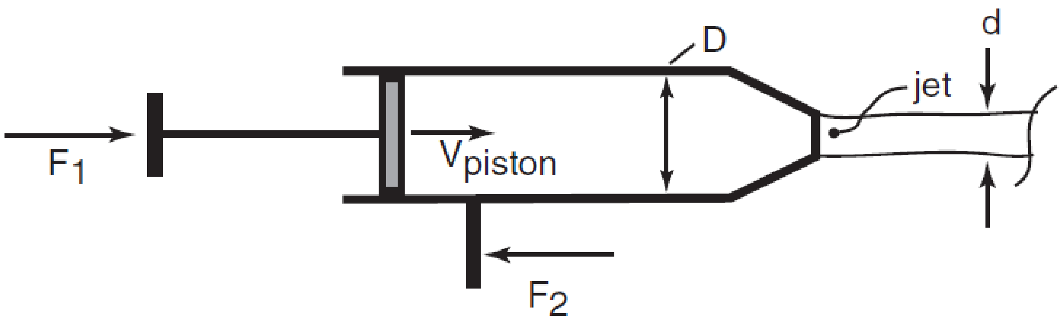
Solution for problem 6.14



For this problem, we use the momentum equation in the form of Equation 6.10 (Ed. 10) (or 6.6 of Ed. 9) of the book:

$$\left(\sum \vec{F}\right)_{ext} = \frac{d(m_{cv}\vec{v}_{cv})}{dt} + \sum_{cs} \dot{m}_o \vec{v_o} - \sum_{cs} \dot{m}_\imath \vec{v_\imath}$$

In this problem, all the vectors are only in one direction (x-direction); we do not have any inlet streams ($\dot{m}_\imath = 0$); there is only one outlet; and we only have two external forces acting on the control volume, namely F_1 and F_2. Thus we have:

$$F_1 - F_2 = \frac{d(m_{cv}V_{cv})}{dt} + \dot{m}_o V_{ox}$$

First we calculate F_1. To do this, consider that the piston is moving with constant velocity (zero acceleration). So the sum of the forces acting on the piston should be equal to zero. There is only two forces acting on the piston: the gage pressure from the fluid inside the cylinder and F_1. Therefore:

$$F_1 = p_{inside}.A_{piston}$$

To find the gage pressure of the fluid inside the cylinder we can apply the Bernoulli equation between exit of the cylinder and inside the cylinder:

$$p_{inside} + \frac{\rho V_{piston}^2}{2} = \frac{\rho V_{out}^2}{2} \implies p_{inside} = \frac{\rho\left(V_{out}^2 - V_{piston}^2\right)}{2}$$

We have the velocity of the piston, so now let's calculate the outlet velocity. To do this we use the continuity equation:

$$\left(\frac{\pi D^2}{4}\right)V_{piston} = \left(\frac{\pi d^2}{4}\right)V_{out} \implies V_{out} = V_{piston}\left(\frac{D^2}{d^2}\right) = \left(0.3\frac{m}{s}\right).\left(\frac{80\ mm}{15\ mm}\right)^2$$

$$V_{out} = 8.53\ m/s$$

Thus, now p_{inside} can be calculated:

$$p_{inside} = \frac{(998\ kg/m^3)((8.53\ m/s)^2 - (0.3\ m/s)^2)}{2} = 36.3\ kPa$$

Now, F_1 can be calculated too:

$$F_1 = p_{inside}.A_{piston} = (36300\ Pa).\left(\frac{\pi(0.08\ m)^2}{4}\right)$$

$$F_1 = 182\ N$$

Now that we have computed all the necessary quantities, we go back to our simplified momentum equation:

$$F_1 - F_2 = \frac{d(m_{cv}V_{cv})}{dt} + \dot{m}_o V_{ox}$$

Here, we expand the derivative $\frac{d(m_{cv}V_{cv})}{dt}$:

$$F_1 - F_2 = m_{cv}\frac{d(V_{cv})}{dt} + V_{cv}\frac{d(m_{cv})}{dt} + \dot{m}_o V_{ox}$$

We can assume all the fluid inside the cylinder is moving with the same velocity as the piston, thus because the piston has a constant velocity, we have $\frac{d(V_{cv})}{dt} = 0$.

From the continuity equation we have:

$$\frac{d(m_{cv})}{dt} = -\dot{m}_o$$

and

$$\dot{m}_o = \rho V_{out}\frac{\pi d^2}{4} = \left(998\ \frac{kg}{m^3}\right)\left(8.53\frac{m}{s}\right)\left(\frac{\pi(0.015\ m)^2}{4}\right) = 1.50\ kg/s$$

Hence, the momentum equation can be further simplified as:

$$F_1 - F_2 = \dot{m}_o(V_{ox} - V_{cv})$$

where $V_{ox} = V_{out}$ and $V_{cv} = V_{piston}$.

Thus:

$$F_2 = F_1 - \dot{m}_o\left(V_{out} - V_{piston}\right) = 182\ N - (1.50\ kg/s)\left(8.53\frac{m}{s} - 0.3\ \frac{m}{s}\right)$$

$$F_2 = 170\ N$$

So, the magnitude of F_1 is larger than F_2.

6.23: PROBLEM DEFINITION

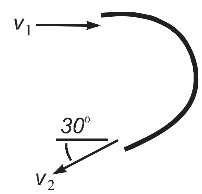
Situation:

A fixed vane in the horizontal plane.
$v_1 = 22$ m/s, $v_2 = 21$ m/s.
$Q = 0.15$ m^3/s, $S = 0.9$.

Find:

Components of force to hold vane stationary (kN).

Sketch:

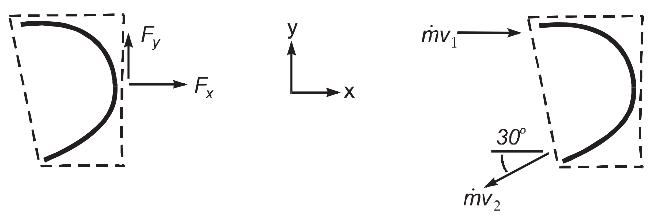


PLAN

Apply the momentum equation.

SOLUTION

Force and momentum diagrams



Mass flow rate

$$\begin{aligned}\dot{m} &= \rho Q \\ &= 0.9 \times 1000\,\mathrm{kg/\,m^3} \times 0.15\,\mathrm{m^3/\,s} \\ &= 135 \text{ kg/s}\end{aligned}$$

Momentum equation (x-direction)

$$\begin{aligned}\sum F_x &= \dot{m}\,(v_o)_x - \dot{m}\,(v_i)_x \\ F_x &= \dot{m}(-v_2 \cos 30^\circ) - \dot{m}v_1 \\ F_x &= -135\,\mathrm{kg/\,s}(21\,\mathrm{m/\,s}\cos 30^\circ + 22\,\mathrm{m/\,s})\end{aligned}$$

$$\boxed{F_x = -5.43\,\mathrm{kN}\ \text{(acts to the left)}}$$

Momentum equation (y-direction)

$$\begin{aligned} \sum F_y &= \dot{m}\,(v_o)_y - \dot{m}\,(v_i)_y \\ F_y &= \dot{m}\,(-v_2 \sin 30^\circ) \\ &= 135\,\mathrm{kg/\,s}\,(-21\,\mathrm{m/\,s} \sin 30^\circ) \\ &= -1.42\ \mathrm{kN} \end{aligned}$$

$$\boxed{F_y = -1.42\,\mathrm{kN}\ \text{(acts downward)}}$$

Solution for problem 6.35

As was mentioned in the class, this problem was corrected in a way that the diameter of each of the two outgoing streams is given as $\frac{D}{\sqrt{2}}$, where D is the diameter of the incoming stream.

Select a control volume surrounding the moving cone. Select a reference frame fixed to the cone. Section 1 is the inlet. Section 2 is the outlet. Inlet velocity (relative to the reference frame and surface of the control volume) is:

$$v_1 = V_1 = (60+5)\frac{m}{s} = 65\frac{m}{s}$$

Then, we use the continuity equation to find the (relative) velocity of the two outlet streams. Considering we have two outgoing streams with a relative velocity of V_2 and with a diameter of $\frac{D}{\sqrt{2}}$, we have:

$$\frac{\pi D^2}{4}\,|V_1| = \frac{\pi\left({}^{D}\!/_{\sqrt{2}}\right)^2}{4}\,|V_2| \times 2 = \frac{\pi D^2}{4}\,|V_2| \quad \Longrightarrow \; |V_1| = |V_2| = 65\, m/s$$

Since the reference frame is attached to our control volume (and so to our control surface) we have:

$$\vec{v_2} = \vec{V_2}$$

Here, it should be noted that the Bernoulli equation cannot be applied with relative velocities with respect to a moving reference frame.

Momentum equation (x-direction)

$$\begin{aligned} F_x &= \dot{m}(v_{2x} - v_1) \\ &= \rho A_1 V_1 \left(v_2 \cos\theta - v_1\right) \\ &= \rho A_1 V_1^2 \left(\cos\theta - 1\right) \\ &= \left(1000\frac{\text{kg}}{\text{m}^3}\right) \times \left(\frac{\pi \times (0.1\,\text{m})^2}{4}\right) \times (65\,\text{m/s})^2 \left(\cos 50^o - 1\right) \\ &= -11.85 \text{ kN} \end{aligned}$$

$$\boxed{F_x = 11.85 \text{ kN (acting to the left)}}$$

6.62: PROBLEM DEFINITION

Situation:

A 90° pipe bend is described in the problem statement.

Find:

Force on the upstream flange to hold the bend in place.

PLAN

Apply the momentum equation.

SOLUTION

Velocity calculation

$$v = \frac{Q}{A} = \frac{0.34\,\mathrm{m^3/\,s}}{\pi/4 \times (0.3\,\mathrm{m})^2} = 4.8 \text{ m/s}$$

Momentum equation (x-direction)

$$\begin{aligned}
\sum F_x &= \sum_{cs} \dot{m}_o v_{ox} - \sum_{cs} \dot{m}_i v_{ix} \\
pA + F_x &= \rho Q(0 - v) \\
F_x &= 1000\,\mathrm{kg/m^3} \times 0.34\,\mathrm{m^3/\,s}(0 - 4.8\,\mathrm{m/s}) - 28,000\,\mathrm{N/m^2} \times \frac{\pi}{4} \times (0.3\,\mathrm{m})^2 = -3610 \text{ N}
\end{aligned}$$

y-direction

$$\begin{aligned}
F_y &= \rho Q(-v - 0) \\
F_y &= -(1000\,\mathrm{kg/\,m^3} \times 0.34\,\mathrm{m^3/\,s} \times 4.8\,\mathrm{m/\,s}) = -1632 \text{ N}
\end{aligned}$$

z-direction

$$\sum F_z = 0$$

$$-445\,\mathrm{N} - 0.1\,\mathrm{m^3} \times 9810\,\mathrm{N/\,m^3} + F_z = 0$$

$$F_z = +1426 \text{ N}$$

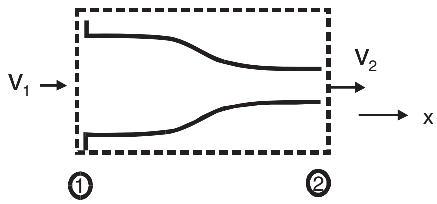
The force is

$$\boxed{\mathbf{F} = (-3610\mathbf{i} \ - 1632\mathbf{j} \ + 1426\mathbf{k}) \text{ N}}$$

6.45: PROBLEM DEFINITION

Situation:

Water flows through a converging nozzle—additional details are provided in the problem statement.



Find:

Force at the flange to hold the nozzle in place: F

PLAN

Apply the Bernoulli equation to establish the pressure at section 1, and then apply the momentum equation to find the force at the flange.

SOLUTION

Continuity equation (select a control volume that surrounds the nozzle).

$$Q_1 = Q_2 = Q = 0.56\,\mathrm{m}^3/\,\mathrm{s}$$

Flow rate equations

$$\begin{aligned} v_1 &= \frac{Q}{A_1} = \frac{4\times Q}{\pi D_1^2} = \frac{4\times(0.56\,\mathrm{m}^3/\,\mathrm{s})}{\pi\,(0.65\,\mathrm{m})^2} \\ &= 1.7\ \mathrm{m/s} \\ v_2 &= \frac{Q}{A_2} = \frac{4\times Q}{\pi D_2^2} = \frac{4\times(0.56\,\mathrm{m}^3/\,\mathrm{s})}{\pi\,(0.225\,\mathrm{m})^2} \\ &= 14.1\,\mathrm{m/s} \end{aligned}$$

Bernoulli equation

$$\begin{aligned} p_1 + \frac{\rho v_1^2}{2} &= p_2 + \frac{\rho v_2^2}{2} \\ p_1 &= 0 + \frac{\rho(v_2^2 - v_1^2)}{2} \\ &= \frac{1000\,\mathrm{kg/m}^3((14.1\,\mathrm{m/s})^2 - (1.7\,\mathrm{m/s})^2)}{2} \\ &= 97,960\,\mathrm{N/m}^2 \end{aligned}$$

Momentum equation (x-direction)

$$p_1 A_1 + F = \dot{m} v_2 - \dot{m} v_1$$

Calculations

$$\begin{aligned} p_1 A_1 &= (97,960\,\mathrm{N/m^2})(\pi/4)(0.65\,\mathrm{m})^2 \\ &= 32,490\,\mathrm{N} \\ \dot{m}v_2 - \dot{m}v_1 &= \rho Q\,(v_2 - v_1) \\ &= (1000\,\mathrm{kg/m^3})(0.56\,\mathrm{m^3/\,s})\,(14.1 - 1.7)\ \mathrm{m/s} \\ &= 6944\,\mathrm{N} \end{aligned}$$

Substituting numerical values into the momentum equation

$$\begin{aligned} F &= -p_1 A_1 + (\dot{m}v_2 - \dot{m}v_1) \\ &= -32,490\,\mathrm{N} + 6944\,\mathrm{N} \\ &= -25,546\,\mathrm{N} \end{aligned}$$

$$\boxed{F = -25,546\ \ \mathrm{N}\ \text{(acts to left)}}$$

6.78: PROBLEM DEFINITION

Situation:

A flow in a pipe is laminar and fully developed—additional details are provided in the problem statement.

Find:

Derive a formula for the resisting shear force (F_τ) as a function of the parameters $D,\ p_1, p_2,\ \rho$, and U.

PLAN

Apply the momentum equation, then the continuity equation.

SOLUTION

Momentum equation (x-direction)

$$\begin{aligned}
\sum F_x &= \int_{cs} \rho v(v \cdot dA) \\
p_1A_1 - p_2A_2 - F_\tau &= \int_{A_2} \rho u_2^2 dA - (\rho A u_1)u_1 \\
p_1A - p_2A - F_\tau &= -\rho u_1^2 A + \int_{A_2} \rho u_2^2 dA \qquad (1)
\end{aligned}$$

Integration of momentum outflow term

$$\begin{aligned}
u_2 &= u_{\max}(1-(r/r_0)^2)^2 \\
u_2^2 &= u_{\max}^2(1-(r/r_0)^2)^2 \\
\int_{A_2} \rho u_2^2 dA &= \int_0^{r_0} \rho u_{\max}^2 (1-(r/r_0)^2)^2 2\pi r dr \\
&= -\rho u_{\max}^2 \pi r_0^2 \int_0^{r_0} (1-(r/r_0)^2)^2(-2r/r_0^2)dr
\end{aligned}$$

To solve the integral, let

$$u = 1 - \left(\frac{r}{r_o}\right)^2$$

Thus

$$du = \left(-\frac{2r}{r_o^2}\right) dr$$

The integral becomes

$$\begin{aligned}
\int_{A_2} \rho u_2^2 dA &= -\rho u_{\max}^2 \pi r_0^2 \int_1^0 u^2 du \\
&= -\rho u_{\max}^2 \pi r_0^2 \left(\frac{u^3}{3} |_1^0 \right) \\
&= -\rho u_{\max}^2 \pi r_0^2 \left(0 - \frac{1}{3} \right) \\
&= \frac{+\rho u_{\max}^2 \pi r_0^2}{3} \qquad (2)
\end{aligned}$$

Continuity equation

$$\begin{aligned}
UA &= \int u dA \\
&= \int_0^{r_0} u_{\max}(1 - (r/r_0)^2) 2\pi \text{ rdr} \\
&= -u_{\max} \pi r_0^2 \int_0^{r_0} (1 - (r/r_0)^2)(-2r/r_0^2) \text{ dr} \\
&= -u_{\max} \pi r_0^2 (1 - (r/r_0)^2)^2 / 2|_0^{r_0} \\
&= u_{\max} \pi r_0^2 / 2
\end{aligned}$$

Therefore

$$u_{\max} = 2U$$

Substituting back into Eq. 2 gives

$$\int_{A_2} \rho u_2^2 dA = 4\rho U^2 \pi r_0^2 / 3$$

Finally substituting back into Eq. 1, and letting $u_1 = U$, the shearing force is given by

$$\boxed{F_\tau = \frac{\pi D^2}{4} \left[p_1 - p_2 - (1/3)\rho U^2 \right]}$$

6.91: PROBLEM DEFINITION

Situation:

A tank of water rests on a sled—additional details are provided in the problem statement.

Find:

Acceleration of sled at time t

PLAN

Apply the momentum equation.

SOLUTION

This type of problem is directly analogous to the rocket problem except that the weight does not directly enter as a force term and $p_e = p_{\text{atm}}$. Therefore, the appropriate equation is

$$\begin{aligned} M\ dv_s/dt &= \rho v_e^2 A_e - F_f \\ a &= (1/M)(\rho v_e^2 (\pi/4) d_e^2 - \mu W) \end{aligned}$$

where μ =coefficient of sliding friction and W is the weight

$$\begin{aligned} W &= 350 + 0.1 \times 1000 \times 9.81 = 1331 \text{ N} \\ a &= (g/W)(1,000 \times 25^2 (\pi/4) \times 0.015^2 - (1331 \times 0.05)) \\ &= (9.81/1,331)(43.90) \text{ m/s}^2 \\ & \boxed{a = 0.324 \text{ m/s}^2} \end{aligned}$$

6.93: PROBLEM DEFINITION

Situation:

A water jet ($\rho = 1000\,\mathrm{kg/\,m^3}$) accelerates a cart
$Q = 0.1\ \mathrm{m^3/s}$
Jet speed: $v_j = 10$ m/s.
Cart Mass $M = 10$ kg
Deflection of the jet is normal to the cart.

Find:

(a) Develop an expression for the acceleration of the cart.
(b) Calculate the acceleration when $v_c = 5$ m/s.

Assumptions:

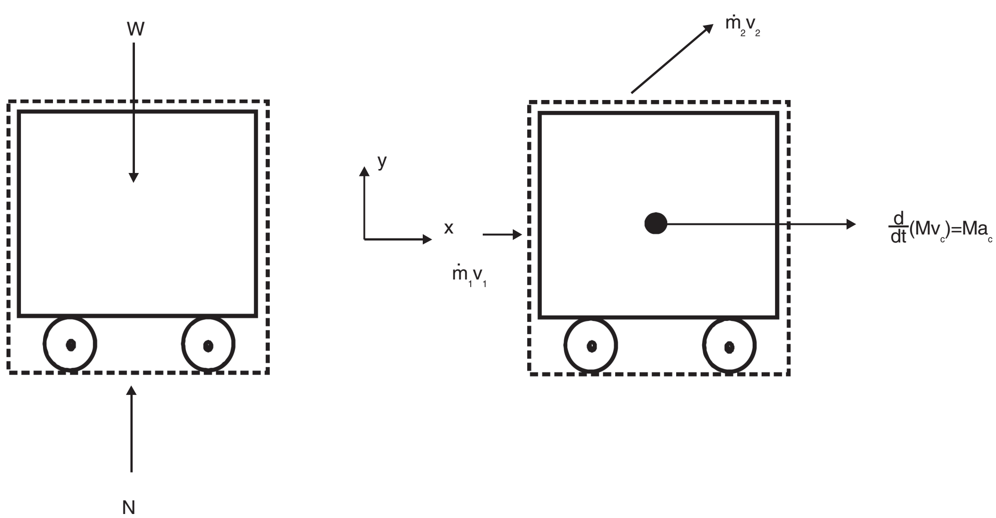
Neglect rolling resistance.
Mass of water << mass of cart.

PLAN

To develop an equation for acceleration of the cart, apply the momentum equation to a cv surround the cart. Select a inertial reference frame fixed to the ground because the cart is accelerating. Then, use continuity and other equations to solve for the acceleration.

SOLUTION

1. Force and momentum diagrams



2. Momentum equation (x-direction)

$$\begin{aligned} \sum F_x &= \frac{d}{dt}(Mv_c) + \dot{m}_2 v_{2_x} - \dot{m}_1 v_1 \\ 0 &= Ma_c + \dot{m}_2 v_{2_x} - \dot{m}_1 v_1 \end{aligned} \tag{1}$$

3. Continuity equation.

$$\dot{m}_2 = \dot{m}_1 = \dot{m} \tag{2}$$

4. Flow Rate Eqn (for flow crossing cs, use velocity relative to cs)

$$\begin{aligned} \dot{m} &= \rho A_j (v_j - v_c) \\ &= \rho \left(\frac{Q}{v_j} \right) (v_j - v_c) \end{aligned} \tag{3}$$

5. Velocity analysis (velocity is relative to fixed reference frame)

$$v_1 = v_j \tag{4a}$$

$$v_{2_x} = v_c \tag{4b}$$

6. Combine Eqs (1) to (4)

$$\begin{aligned} 0 &= M a_c + \dot{m} (v_{2_x} - v_1) \\ 0 &= M a_c + \left[\left(\frac{\rho Q}{v_j} \right) (v_j - v_c) \right] (v_c - v_j) \end{aligned}$$

Solving for acceleration

$$\boxed{a_c = \frac{\rho Q (v_j - v_c)^2}{v_j M}}$$

7. Calculations

$$a_c = \frac{(1000\,\mathrm{kg/\,m^3})\,(0.1\,\mathrm{m^3/\,s})\,(10\,\mathrm{m/\,s} - 5\,\mathrm{m/\,s})^2}{(10\,\mathrm{m/\,s})\,(10\,\mathrm{kg})}$$

$$\boxed{a_c = 25 \ \ \mathrm{m/\,s^2} \ \ \ (\text{when } v_c = 5\,\mathrm{m/\,s})}$$

6.65: PROBLEM DEFINITION

Situation:

Water flows through a 60^o pipe bend and jets out to atmosphere—additional details are provided in the problem statement.

Find:

Magnitude and direction of external force components to hold bend in place.

PLAN

Apply the Bernoulli equation, then the momentum equation.

SOLUTION

Flow rate equation

$$\begin{aligned}\left(\frac{D_2}{D_1}\right)^2 v_2 &= \left(\frac{30\,\text{cm}}{60\,\text{cm}}\right)^2 10\,\text{m/s} = 2.5\,\text{m/s} \\ Q &= A_1 v_1 = \pi \times 0.3\,\text{m} \times 0.3\,\text{m} \times 2.5\,\text{m/s} = 0.707\ \text{m}^3/\text{s}\end{aligned}$$

Bernoulli equation

$$\begin{aligned}p_1 &= p_2 + \frac{\rho}{2}(v_2^2 - v_1^2) \\ &= 0 + \frac{1000\,\text{kg/m}^3}{2}\left(\left(\frac{10\,\text{m}}{\text{s}}\right)^2 - \left(\frac{2.5\,\text{m}}{\text{s}}\right)^2\right) \\ &= 46,875\ \text{Pa gage}\end{aligned}$$

Momentum equation (x-direction)

$$\begin{aligned}F_x + p_1 A_1 &= \rho Q(-v_2 \cos\ 60^\circ - v_1) \\ F_x &= -46,875\,\text{Pa} \times \pi \times 0.3\,\text{m} \times 0.3\,\text{m} \\ &\quad +1000\,\text{kg/m}^3 \times 0.707\,\text{m}^3/\text{s} \times (-10\,\text{m/s}\cos 60^\circ - 2.5\,\text{m/s}) \\ &= -18,560\ \text{N}\end{aligned}$$

y-direction

$$\begin{aligned}F_y &= \rho Q(-v_2 \sin 60^\circ - v_1) \\ F_y &= 1000\,\text{kg/m}^3 \times 0.707 \times (-10\,\text{m/s}\sin 60^\circ - 0) \\ &= -6120\ \text{N}\end{aligned}$$

z-direction

$$\begin{aligned}F_z - W_{\text{H}_20} - W_{\text{bend}} &= 0 \\ F_z &= (0.25\,\text{m}^3 \times 9,810\,\text{N/m}^3) + (250\,\text{kg} \times 9.81\,\text{m/s}^2) = 4,905\ \text{N}\end{aligned}$$

Net force

$$\boxed{\mathbf{F} = (-18.6\mathbf{i} - 6.12\mathbf{j} + 4.91\mathbf{k})\ \text{kN}}$$